

Chaos

for The Institution of Engineering and
Technology

by
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Early history of a new branch of mathematics; chaos

- The study of chaos was enabled by the invention of the computer
 - As early as the 1890s, Poincare' had found some evidence of its existence when studying orbits
- In the early 1960s its implications for weather prediction were found by Ed Lorenz of MIT (**more on him coming up**)
 - The 'butterfly' effect is attributed to Lorenz
- Currently chaos theory remains an area of research involving many different disciplines

- | | |
|--|--|
| <ul style="list-style-type: none">● Mathematics● Topology● Physics● Length of a coast line● Social systems● Population modeling | <ul style="list-style-type: none">● Biology● Meteorology● Astrophysics● Information theory● Computational neuroscience |
|--|--|



The definition of chaos

- A nonlinear but deterministic relationship between input and output
 - Note that although we will see output that looks random, that is not the case
 - Outputs from chaotic systems are deterministic (same result each time but highly sensitive to initial conditions)
 - Lorenz says “**Chaos: When the present determines the future, but the approximate present does not approximately determine the future.**”
- Tiny changes in input can lead to dramatic changes in output
- Why study this?
 - Because nature is inherently non-linear
 - Emphasis in education is on solutions to linear systems but some attention should be paid to more realistic models

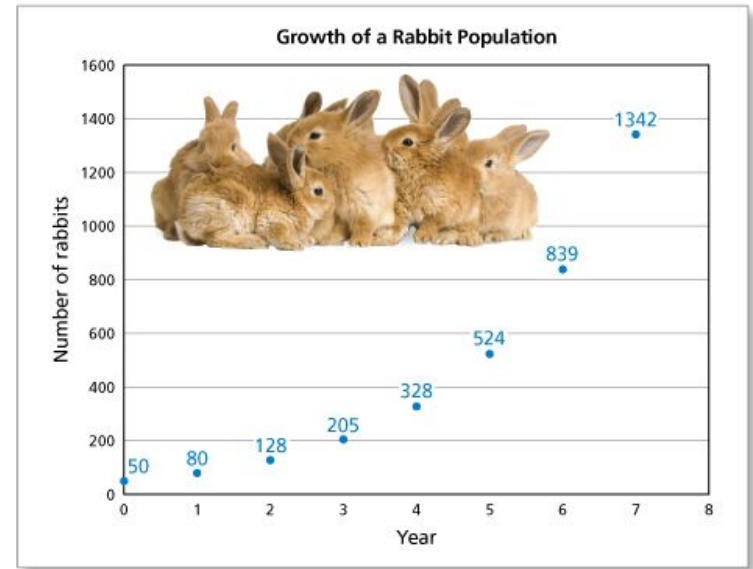
A double rod pendulum animation shows chaotic behavior. Starting the pendulum from a slightly different initial condition would result in a completely different trajectory. The double rod pendulum is one of the simplest dynamical systems that has chaotic solutions.



<https://en.wikipedia.org/wiki/File:Double-compound-pendulum.gif>

The concept of recursion

- Consider a simple system; a rancher with rabbits
- The rabbits produce one pair of offspring per year
- We want to know how many breeding pairs we have in an ideal setting where there are no predators, no disease, plenty of food and water, etc.
- At the end of the first year, we have the original breeding pair, plus the first pair of offspring. So, 2 breeding pairs.
- Each produce one breeding pair in the next season so we now have 4.



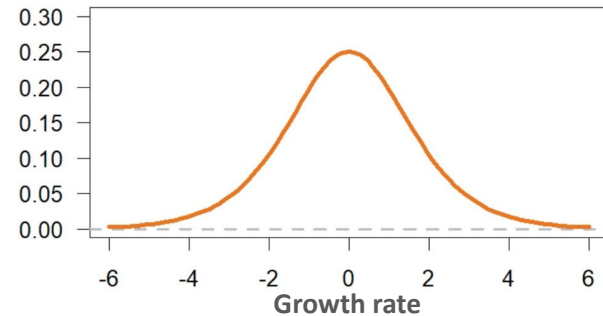
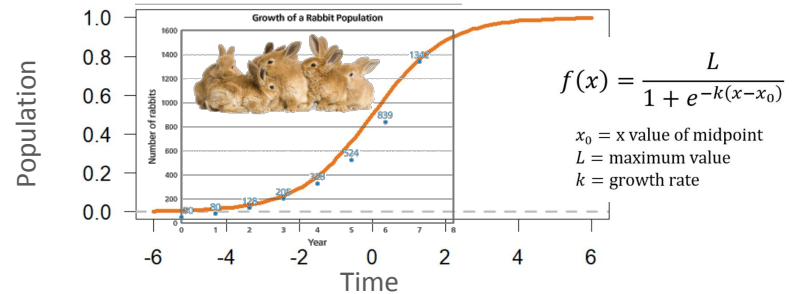
This process continues until the number of rabbits is limited by the available food, predation and other factors that limit number of offspring that will mature and produce their own offspring.

The Logistics Function is widely used to characterize the limits of dynamical systems

- Its simplest form is the so-called sigmoid, first introduced in the early 1800s.
- The logistic function finds applications in a range of fields; in this lecture we focus on population dynamics.
 - Other fields include
 - Spread of disease
 - Heart attacks
 - Weather prediction
- A convenient way to analyze its behavior is to take its derivative (slope) to generate the **logistics map**.

[Logistic function - Wikipedia](#)

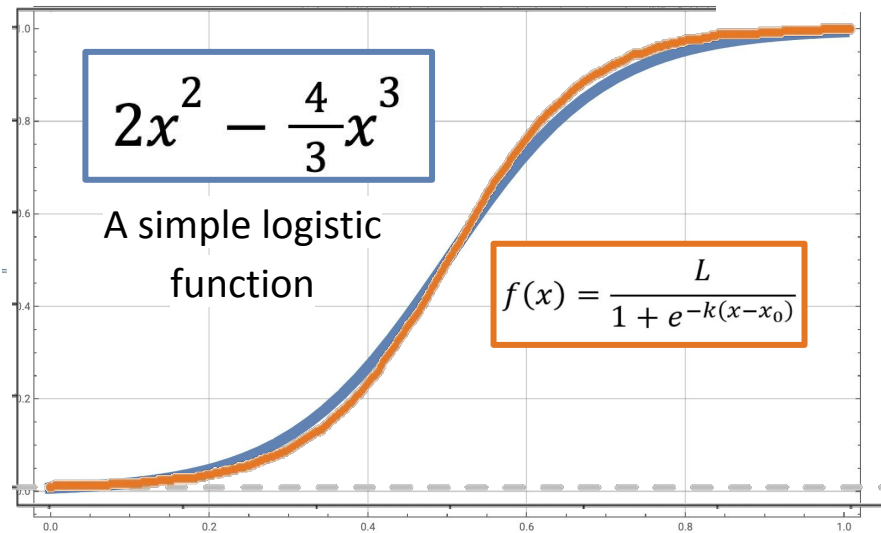
The logistic function conveys a sense of population as a function of time



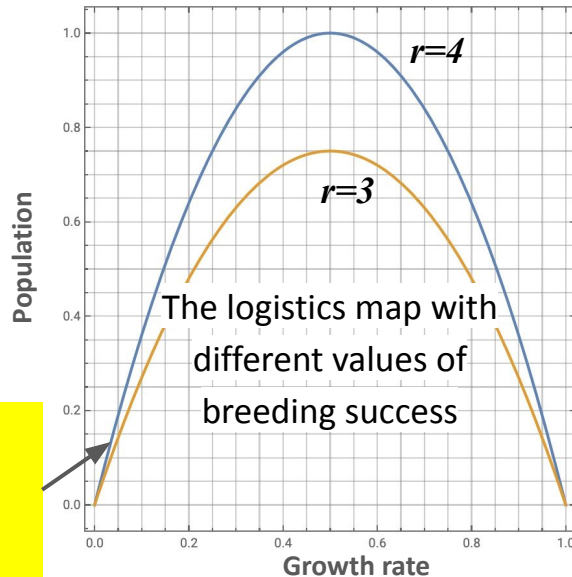
- The **logistics map** captures the rate of increase of population
- It is the derivative of the logistics function
- Low values when the population is low
- Low values when the population is reaching the limits of the environment

A simple version of the logistics function leads to a parabola for the logistics map

$$x_{n+1} = r \cdot x_n \cdot (1 - x_n)$$

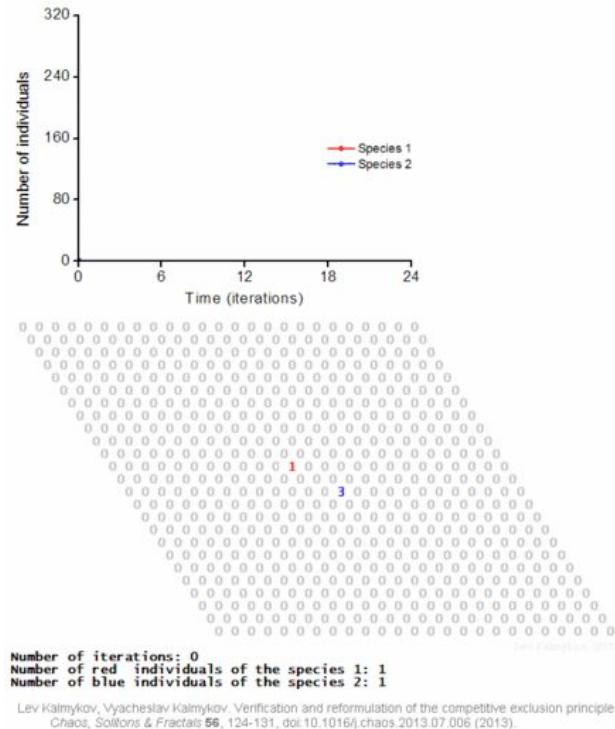
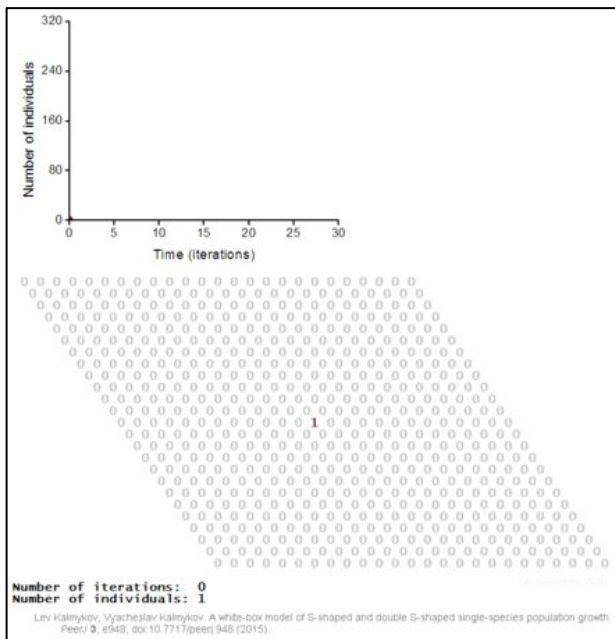


Where the subscripts n indicate each generation of animals

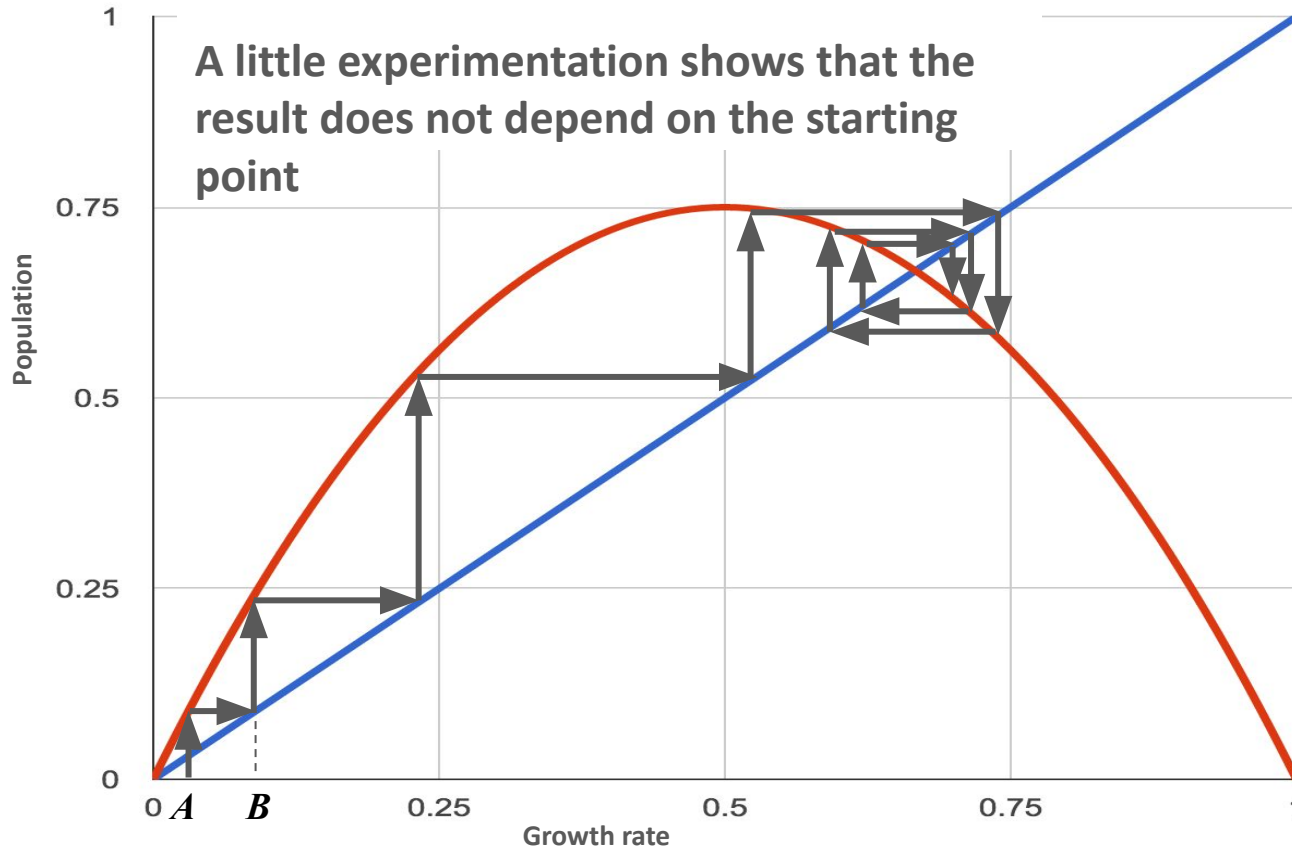


We will see later that virtually any shape like this logistics map results in the same type of behavior. This is known as universality.

The logistics function implies an 's' curve for population as a function of time



Iteration of the logistic map can be demonstrated with two very simple curves; **no math required**

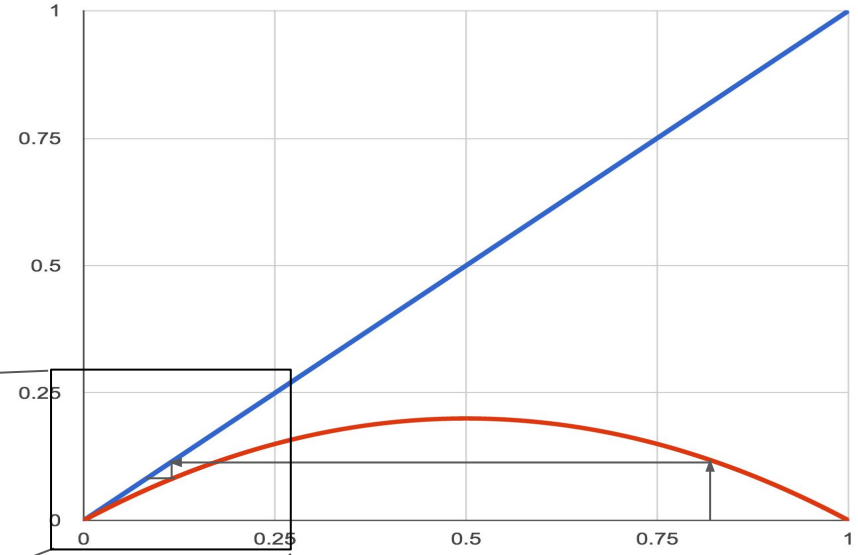
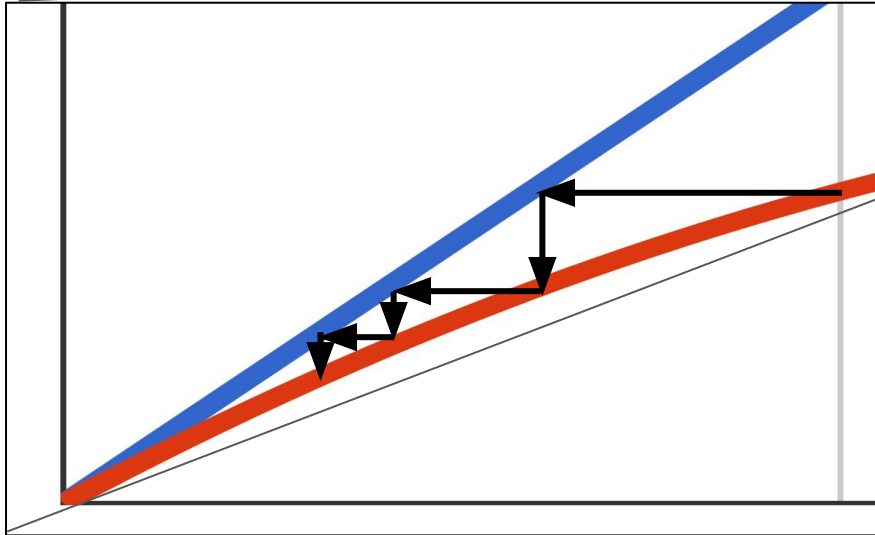


- Start with any number, indicated by $x=A$
- Use the formula for the parabola $y=r x(1-x)$ to compute the value at A
- You could lay down the length of the vector $y(A)$ horizontally and repeat the process BUT it is easier to do this is graphically.
- The diagonal line equates x and y so going horizontally gets us to B
- Now repeat the process

- In this case, $r=3$ and $A=0.04$
- After 10,000 iterations, two values emerge for y ; 0.66431 and 0.66901 eventually settling on 0.666...

When $r < 1$

- The line and curve only intersect at zero, so no matter where we start we are driven to zero
- Here we start around 0.8



Calculus interlude

This can be shown to be true by considering the slope of the curve

$$f(x) = rx(1-x)$$

$$f'(x) = r - 2rx$$

So the maximum slope occurs when $x=0$

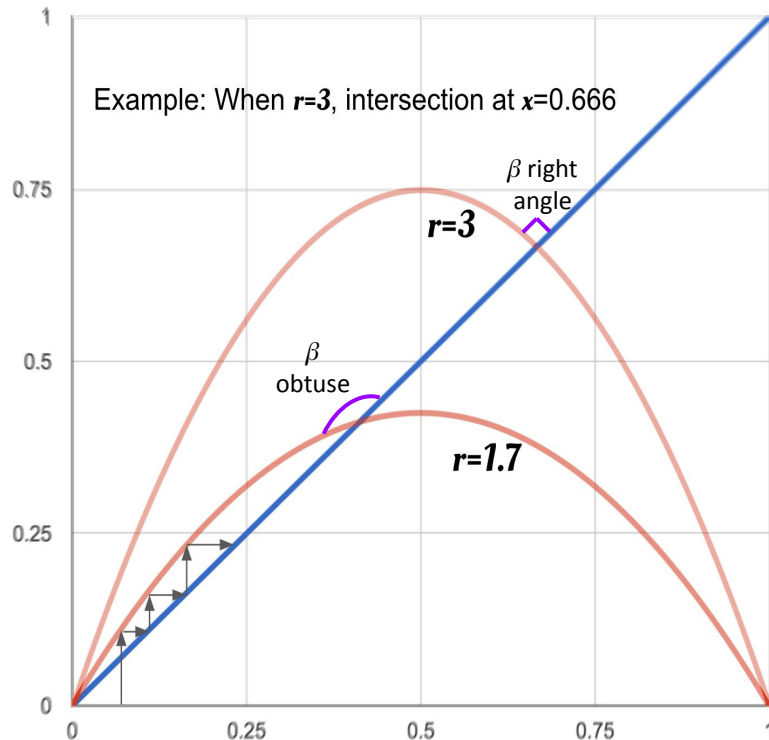
$$f'(x) = r$$

If $r < 1$, $f' < 1$, the curve and the line only touch at $x=0$

If $r > 1$, $f' > 1$ and the line and the curve must intersect at 2 places

When $1 < r < 3$ the iterations take us to the single point of intersection of the line and curve

- The angle β of intersection of the logistics map and the diagonal line are critical to recognizing how the system will evolve.
- Above $r > 3$ things get strange!

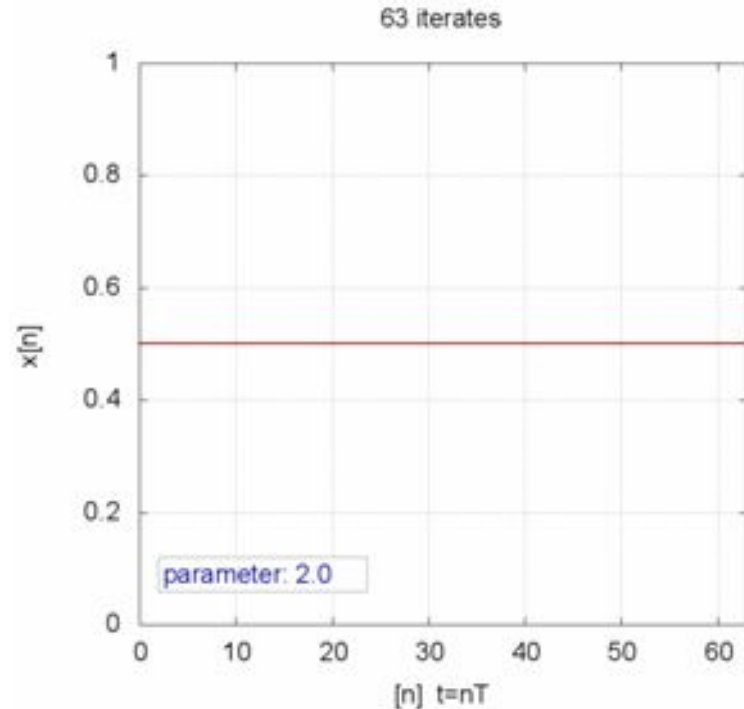
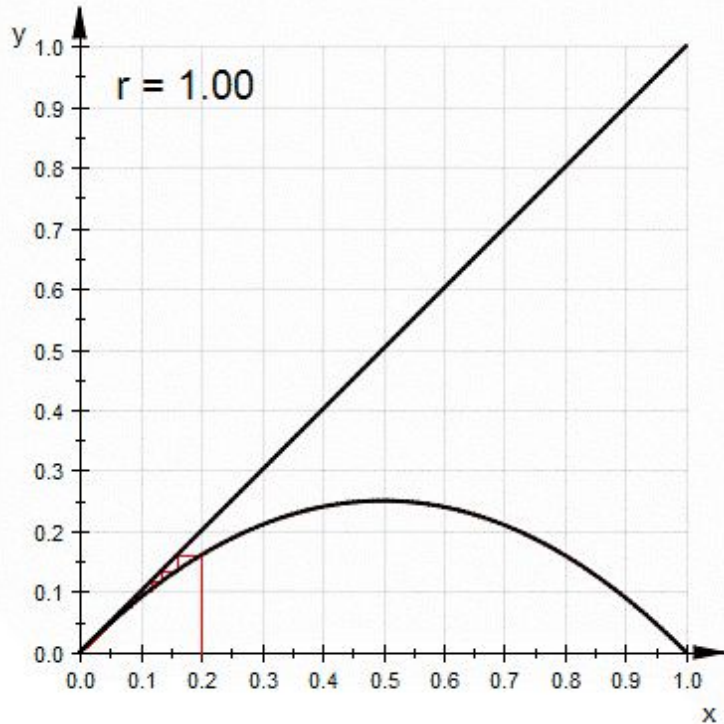


The underlying math for this simple case

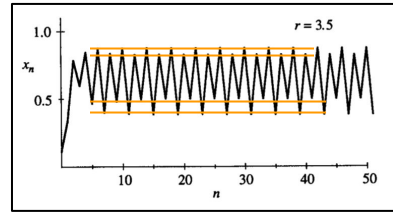
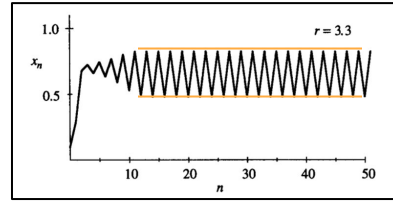
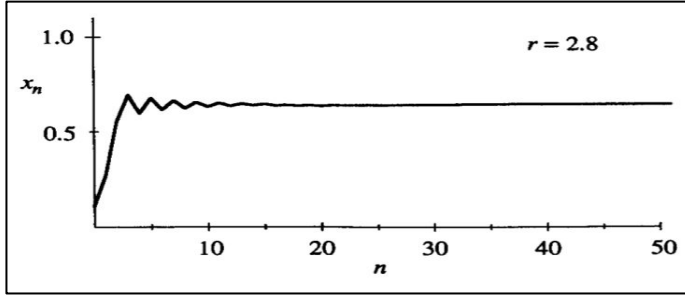
Equation of line \swarrow \nwarrow Equation of curve

$$x = rx(1-x)$$
$$1-x = \frac{1}{r}$$
$$x = 1 - \frac{1}{r}$$

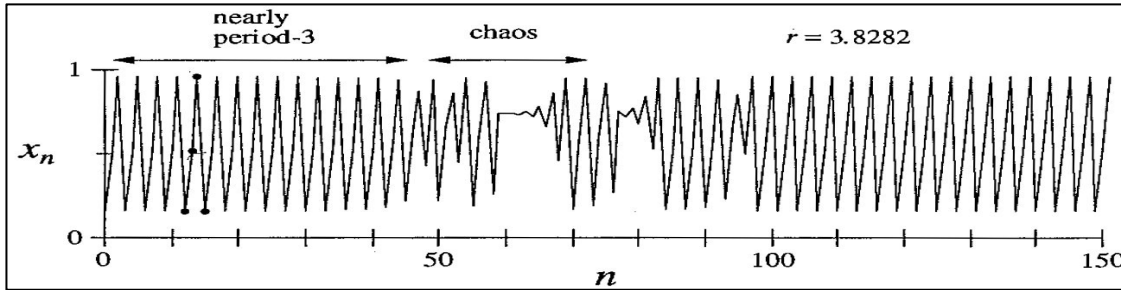
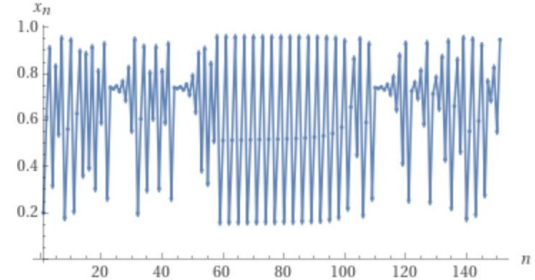
Some animations help show the details



Population vs. time depends on the reproduction rate of the logistics function



[Logistics Function Tool](#)



More explorations of r □

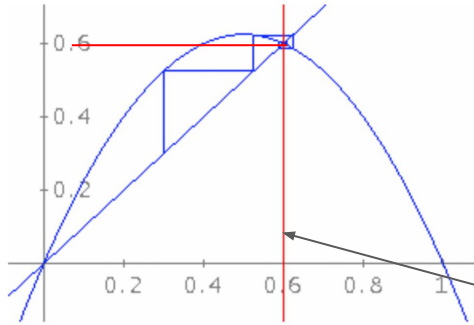


Fig. 1. $r = 2.5$

Red lines indicate that no new values will occur. The growth rate and the population are bounded.

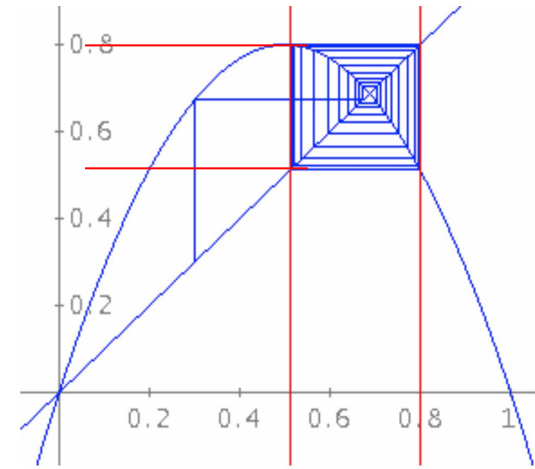


Fig. 2. $r = 3.2$

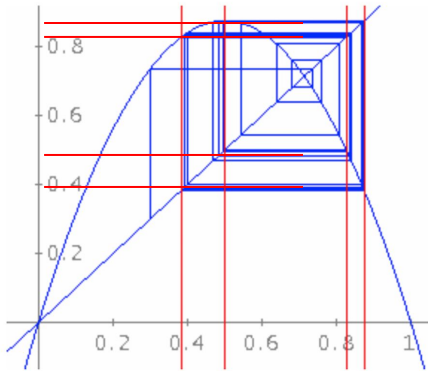
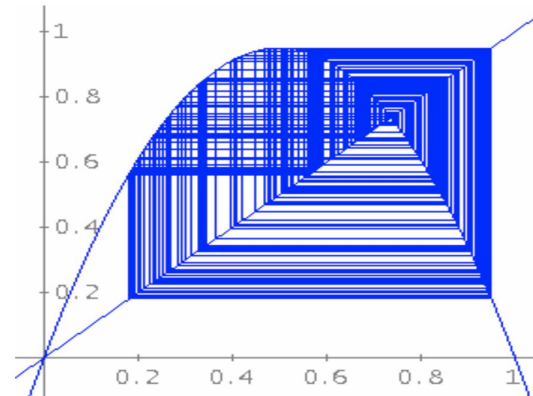


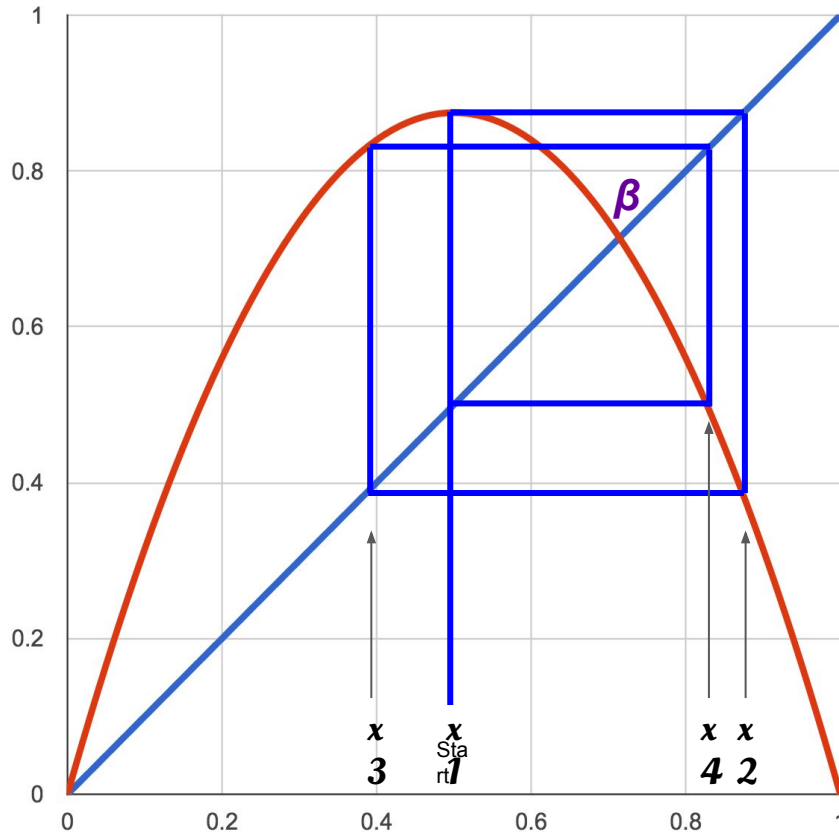
Fig. 3. $r = 3.5$



Note that this case never stops producing new values!

Fig. 4. $r = 3.8$

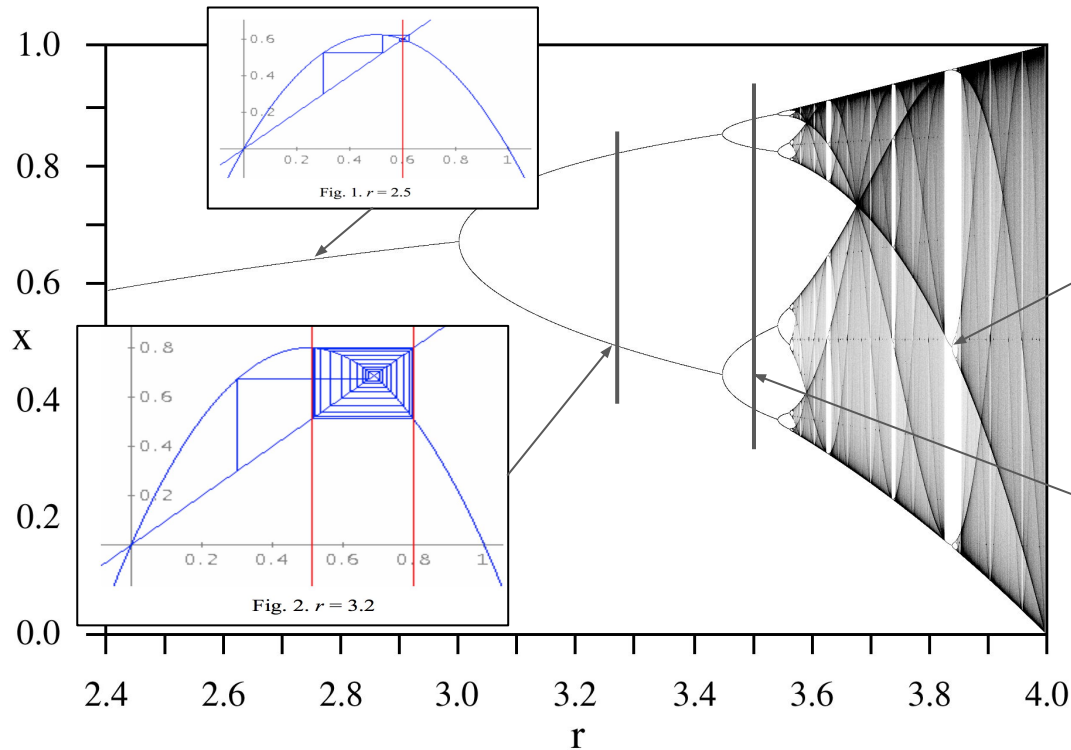
Trajectory for $r > 3$



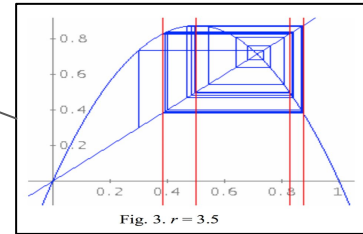
- Finding algebraic solutions for a situation like this is complicated.
- The best approach is numerical solutions
- Here's the basic approach using the logistics map $f(x) = rx(1-x)$
 - $f(x_1) \rightarrow x_2$
 - $f(x_2) \rightarrow x_3$
 - $f(x_3) \rightarrow x_4$
 - $f(x_4) \rightarrow x_1$
- Therefore

$$x_1 = f(f(f(x_1)))$$
- Which requires solving a 4th order polynomial for the 4 values of intersection
- Note that β is now acute

A better way to draw these results is shown below; bifurcation appears at about $r=3$, again at 3.45, again at 3.54.



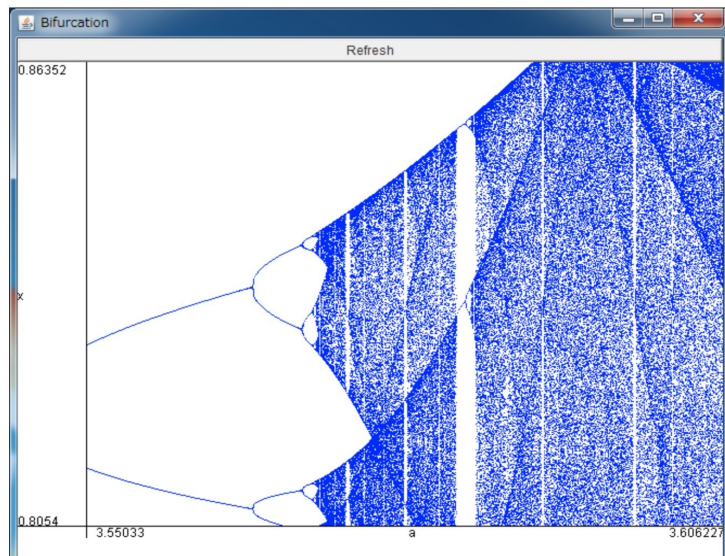
It's hard to see here, but within these white spaces, there repeats of this entire image. More on this later.



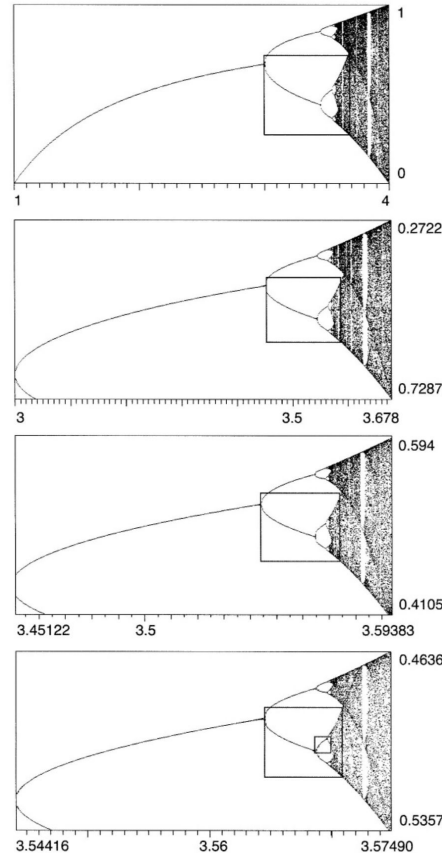
The bifurcation locations are computable with an increasing level of complexity.

We can see more detailed views of the 'white space' by using an online tool

Bifurcation Diagram of Logistic Map



[Bifurcation Diagram of Logistic Map](#)

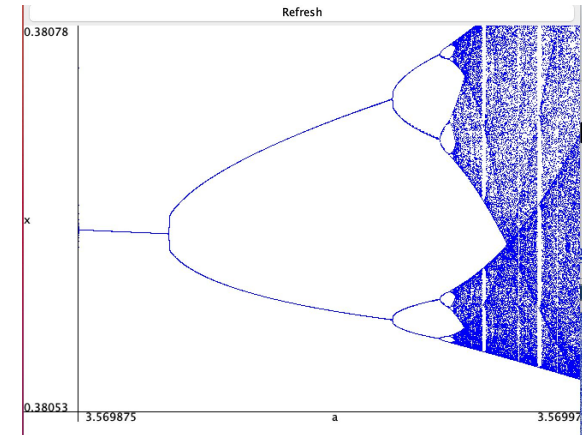


Heinz-Otto Peitgen Hartmut Jürgens Dietmar Saupe

Chaos and Fractals

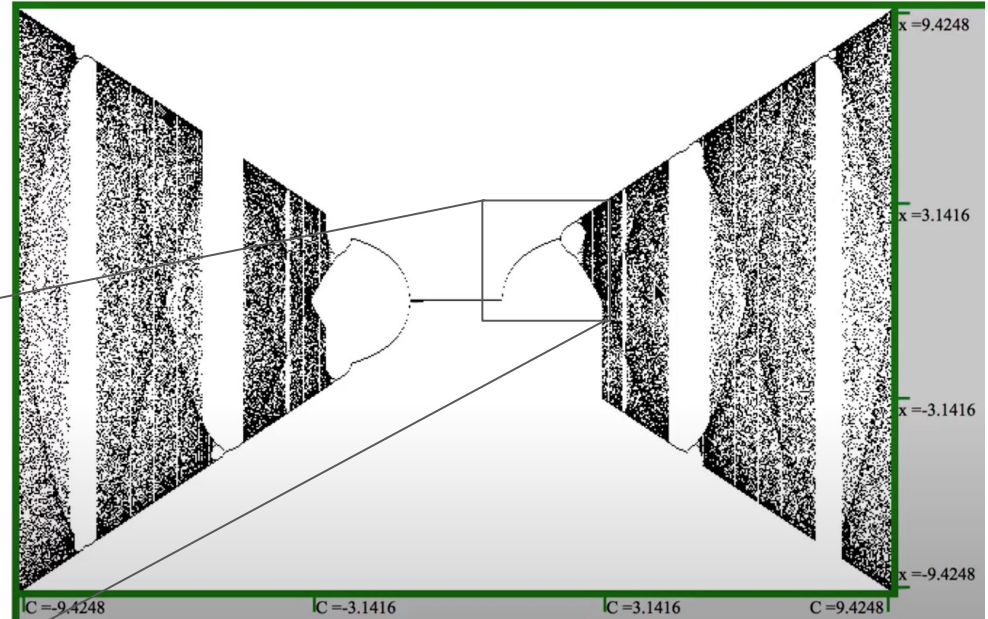
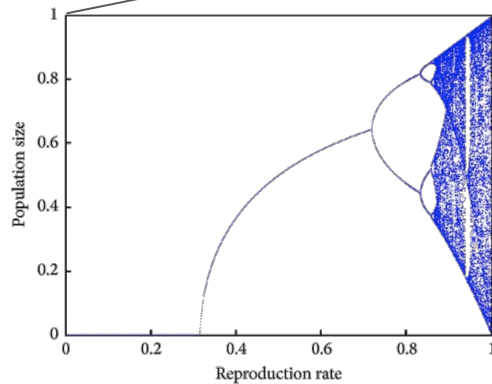
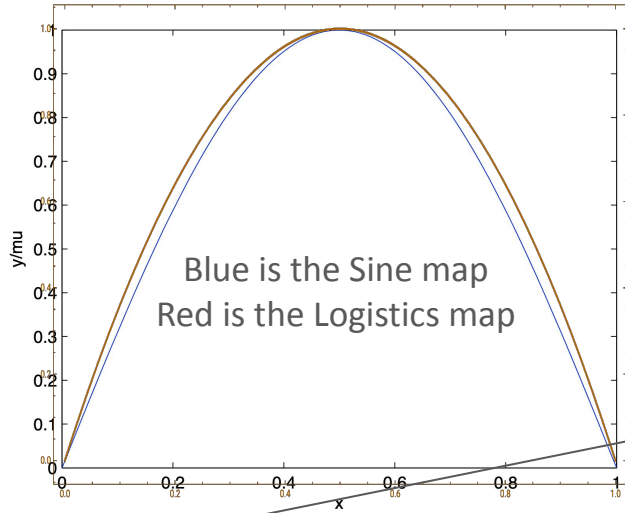
New Frontiers of Science Second Edition

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**Demonstration of
Bifurcation**

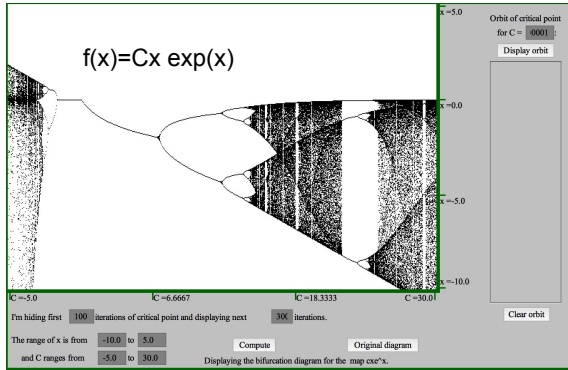
Bifurcation of the sine map



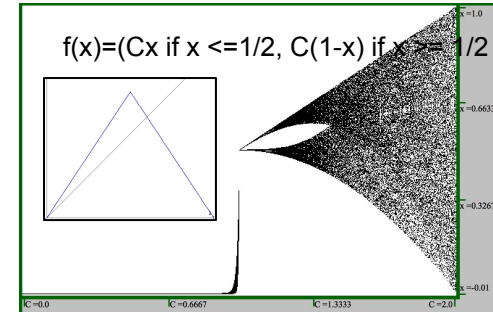
<https://drive.google.com/drive/folders/1PcSZXCSNmrU3t5iNW8xIXRElbnADZpcs>

Bifurcation diagrams for other maps

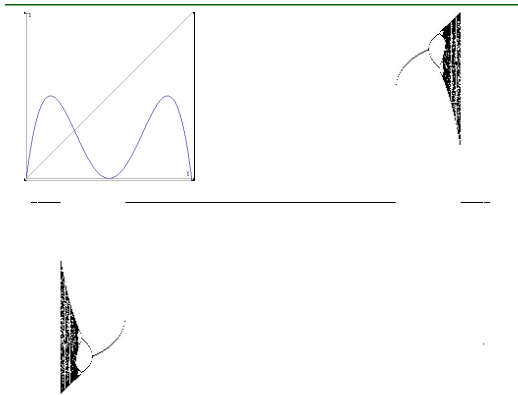
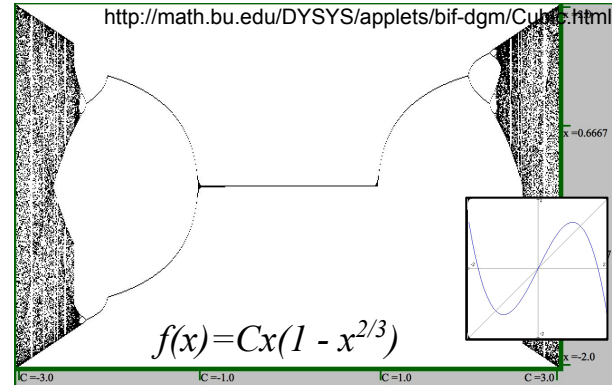
<http://math.bu.edu/DYSYS/applets/bif-dgm/XExp.html>



<http://math.bu.edu/DYSYS/applets/bif-dgm/Tent.html>

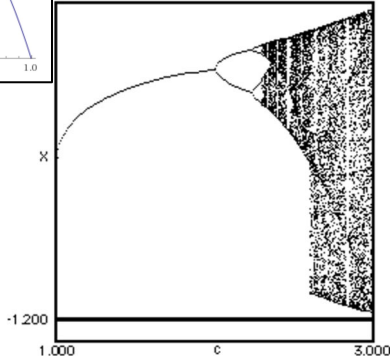
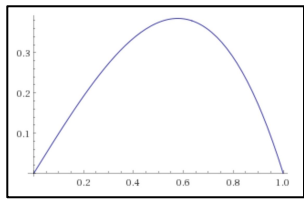


https://upload.wikimedia.org/wikipedia/commons/3/3a/TentMap_BifurcationDiagram.png

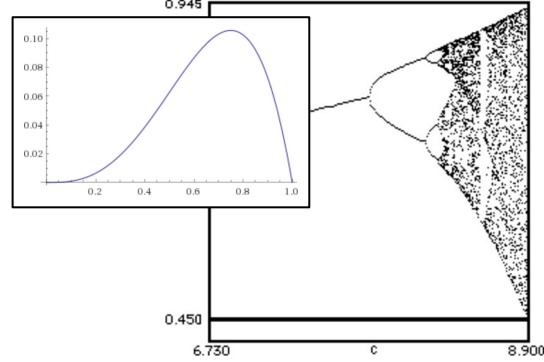


[Orbit Diagram for \$Cx\(2 - x^2\)\$](#)

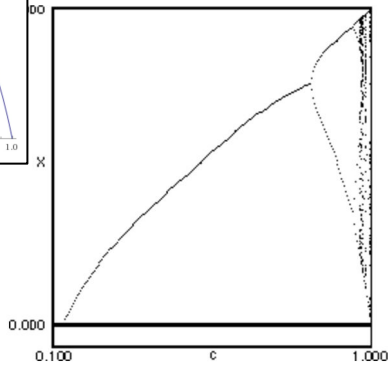
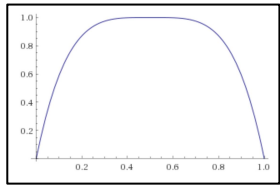
Some more bifurcation diagrams



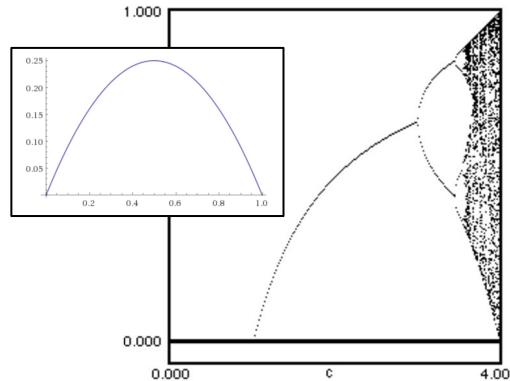
$$x \rightarrow cx(1 - x^2)$$



$$x \rightarrow cx^3(1 - x)$$

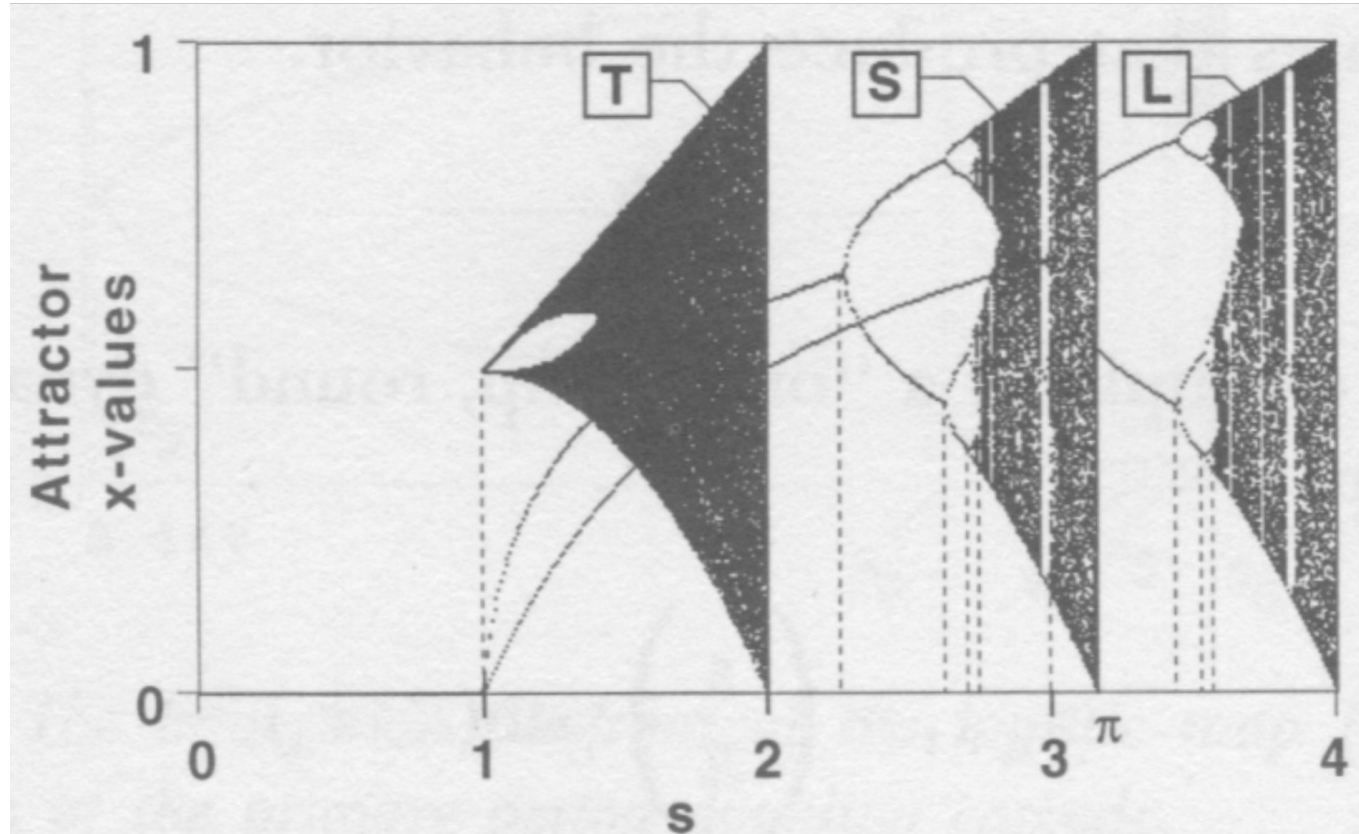


$$x \rightarrow c(1 - (2x - 1)^4)$$



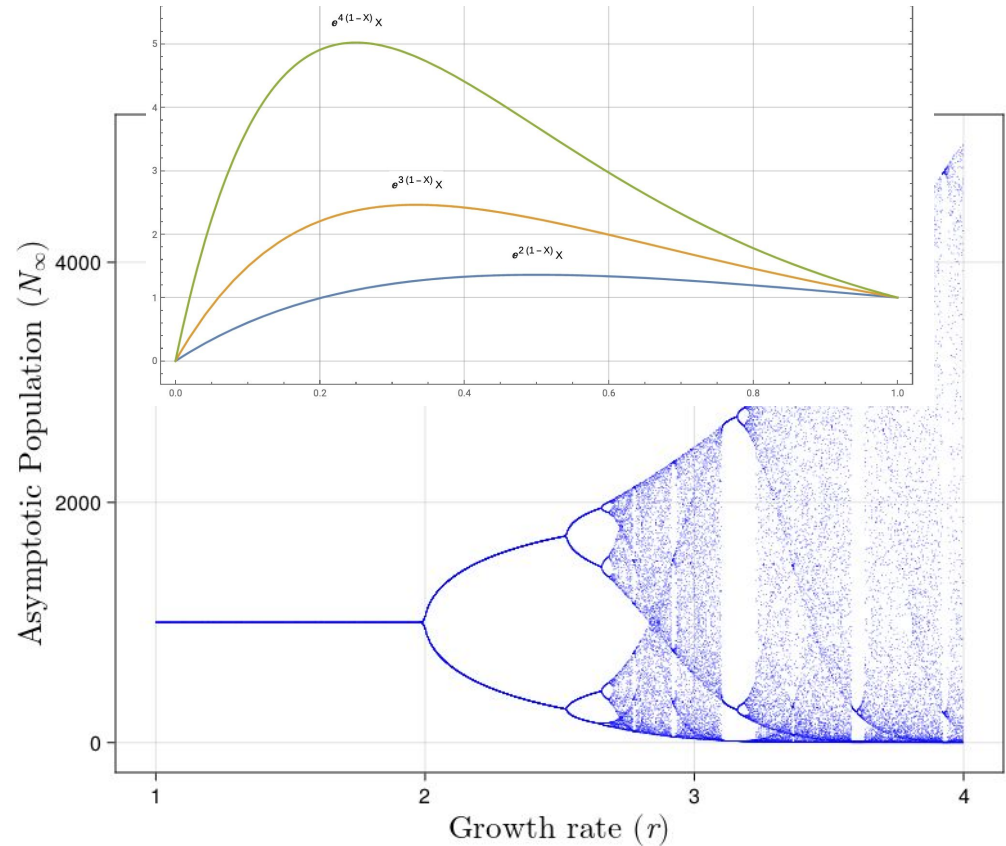
$$x \rightarrow cx(1 - x)$$

Comparison of tent (t), sine (s) and logistics (L) maps



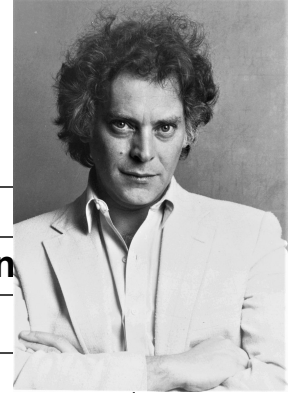
An interesting early example; fish $x_{i+1} = x_i e^{r(1-x_i)}$

- This expression was given by W. E. Ricker (1954). It describes a discrete population model for fishery stocks.
- This model emerged before the modern interpretations of chaos had been invented by Lorenz in 1963.
- A historical note on Ricker's work (https://link.springer.com/chapter/10.1007/1-4020-5366-5_5) mentions his famous fisheries handbook of 1958 ([Handbook of computations for biological statistics of fish populations](#)) but no mention of chaos (still early in its development in the 50s) appears in the book.
- <https://demonstrations.wolfram.com/DiscretePopulationModelForFisheryStocks/#p1>
- [Ricker model - Wikipedia](#)
- <https://demonstrations.wolfram.com/DiscretePopulationModelForFisheryStocks/#p1>



Feigenbaum noticed something called *universality*

$$\delta = \lim_{n \rightarrow \infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}}$$



Example comparison of the logistics function and sine function					
n	Period	Logistics	δ logistics	Sine	δ sine
1	2	3.00000		0.71996	
2	4	3.44949		0.83326	
3	8	3.54409	4.65623	0.85860	4.47119
4	16	3.56441	4.66832	0.86408	4.62409
5	32	3.56876	4.66832	0.86525	4.68376
6	64	3.56969	4.66863		

$$\delta = \lim_{n \rightarrow \infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} = 4.669\ 201\ 609 \dots$$

Feigenbaum's number

The Lorenz water wheel and attractor

- Shows many features of the Lorenz attractor and the role of chaos in weather.
 - [4 Lorenz water wheel.mov](#)
- Phase diagram of the 'other' butterfly
 - Two "chaotic" solutions to the Lorenz system of equations start *almost* the same.
 - Almost no difference (due to nearly identical initial conditions) between the trajectories for time less than 20.
 - The difference is clear for times greater than 25. Trajectories are "chaotic" so it is impossible to predict which equilibria solutions are circling at a given time, t .
 - [5 Lorenz Attractor.mov](#)
 - **Video and most of the discussion is from Prof Jonathan Mitchell, Steven F. Austin State University, Nacogdoches, TX**



The Lorenz attractor equations are deceptively simple

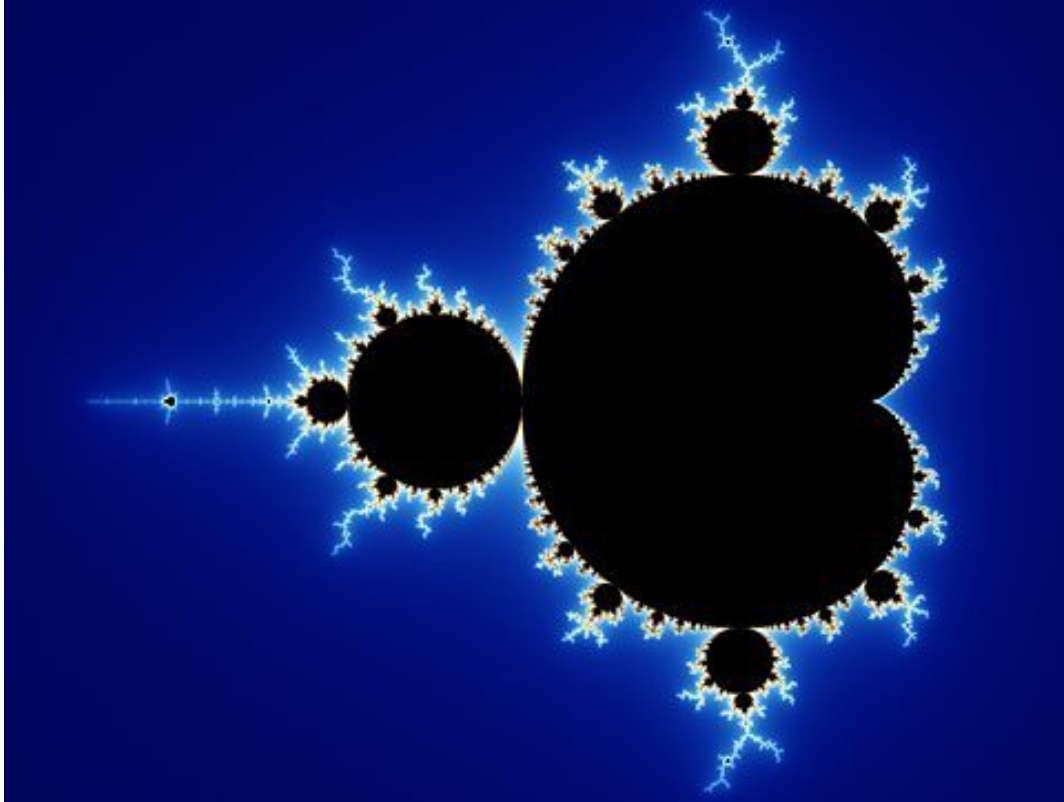
$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x), \\ \frac{dy}{dt} &= x(\rho - z) - y, \\ \frac{dz}{dt} &= xy - \beta z.\end{aligned}$$

σ, ρ, β are the system parameters

[Lorenz system - Wikipedia](#)

The Mandelbrot Set

The Mandelbrot set results from exploring recursive mathematics in the complex plane



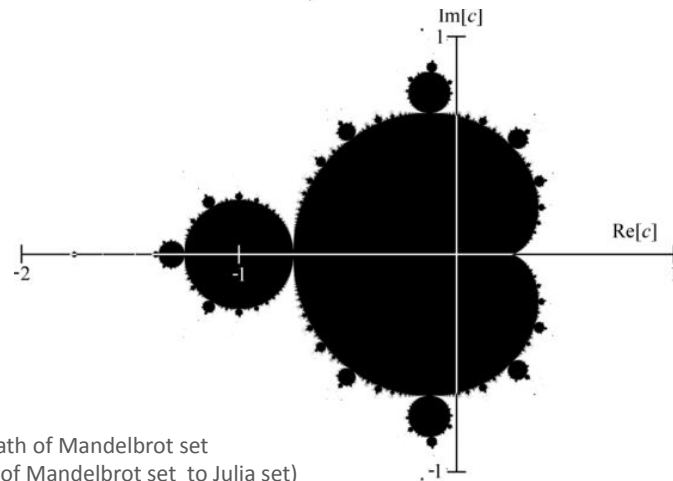
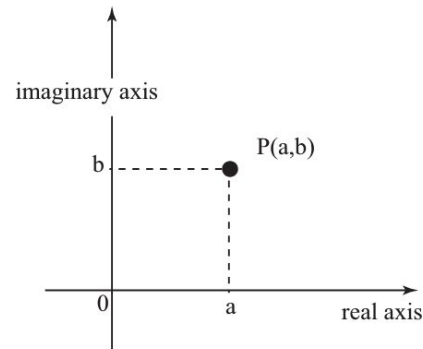
The math is simple but 'complex' since it involves imaginary numbers

- We'll be using complex numbers
- Each point in the complex plane at right represents a complex number
- The Mandelbrot set explores the iterated value of an equation, starting with a first value of zero
- The equation is

$$f(z) = z^2 + C$$

- Some experimentation shows that if C is 1, the values of f increase without bound.
- Similarly $C = -1$ produces results that oscillate between -1 and 0
- **The Mandelbrot set shows the points for which the iterated value never exceeds 2.**

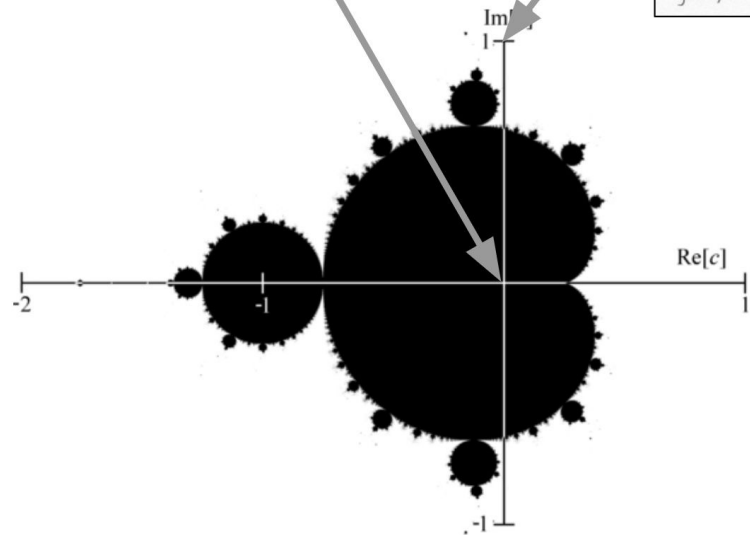
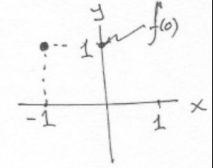
<https://www.youtube.com/watch?v=NGMRB4O922I> (basic math of Mandelbrot set)
<https://www.youtube.com/watch?v=oCkQ7WK7vuY> (relation of Mandelbrot set to Julia set)
<https://www.youtube.com/watch?v=d0vYOCKYhPY> (relation of Mandelbrot set to pi)



Building a Mandelbrot set

$$\begin{aligned} C=0 \\ f(0) &= 0^2 + 0 \\ &= 0 \\ f(0) &= 0 \dots \end{aligned}$$

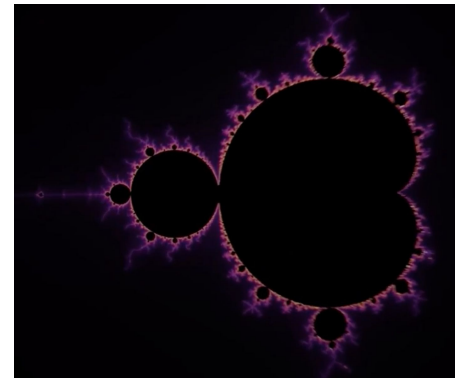
$$\begin{aligned} C=i \\ f(0) &= 0^2 + i = i \\ f(i) &= -1 + i \\ f(-1+i) &= (-1+i)^2 + i = (-1+i)(-1+i) + i \\ &= 1 - i - i + i^2 + i = 1 - 2i - 1 + i = -i \\ f(-i) &= \cancel{(-i)^2 + i} = (-i)^2 + i = -1 + i \end{aligned}$$



Any location on the complex plane bounded by 2 is in the set and appears in the drawing.

The colors dictate how many steps occur before the result either gets bigger than 2 (and runs away) or remains less than 2.

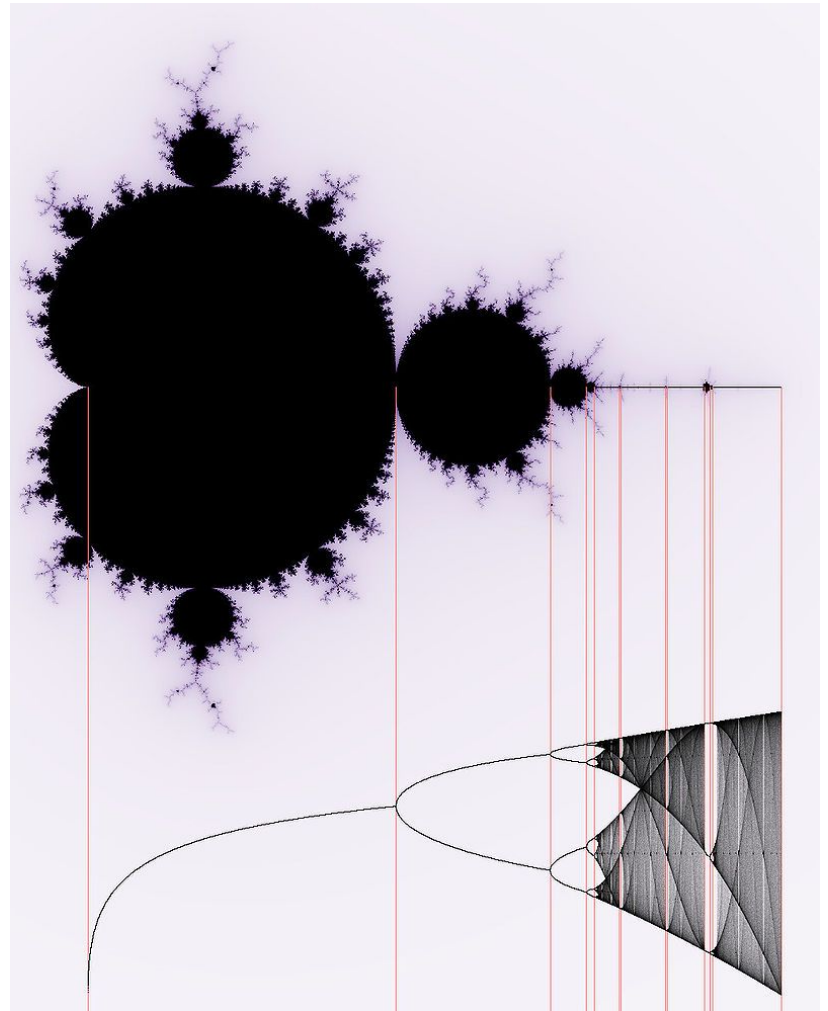
Interesting things happen at the edges. These are places where the set eventually evolves to infinity. Continued 'zooming in' reveals these details.

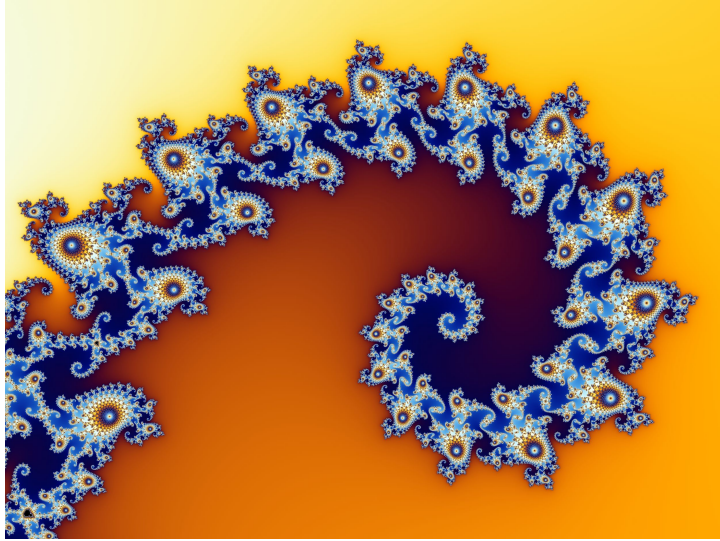


Related points of the logistics function and the Mandelbrot set

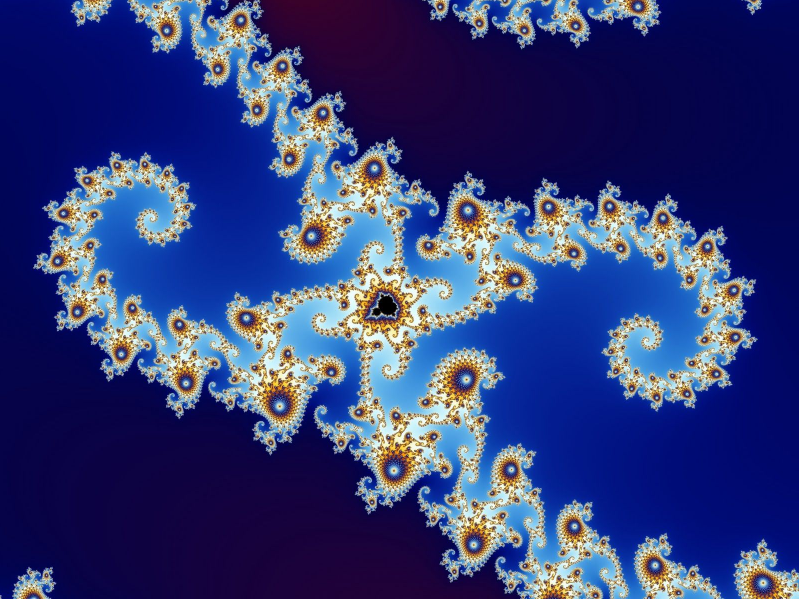
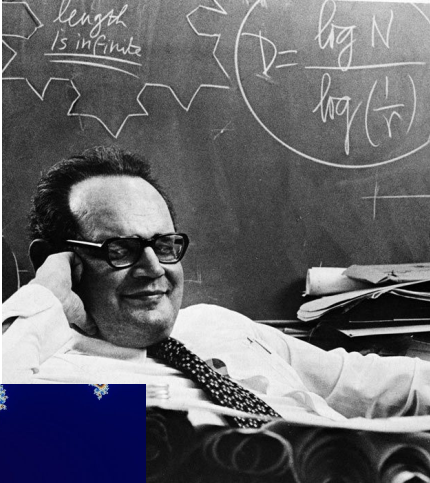
[Logistics and Mandelbrot.mov](#)

[Mandelbrot zoom in](#)

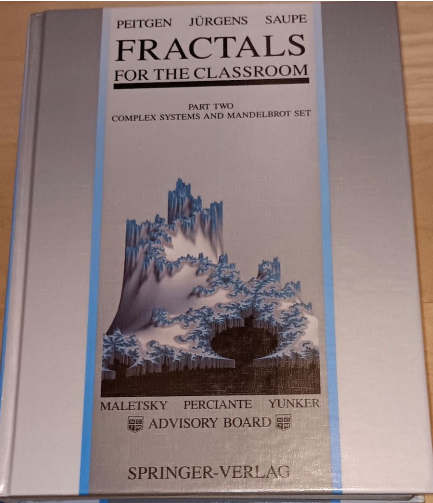
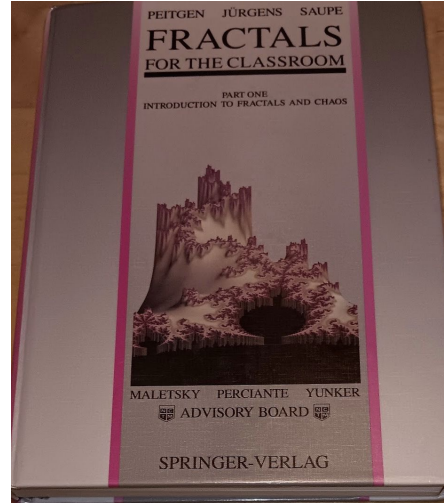
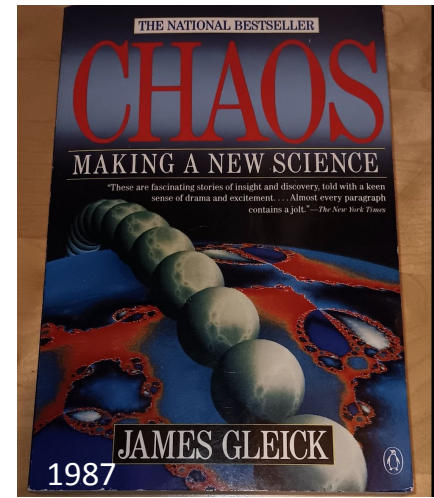
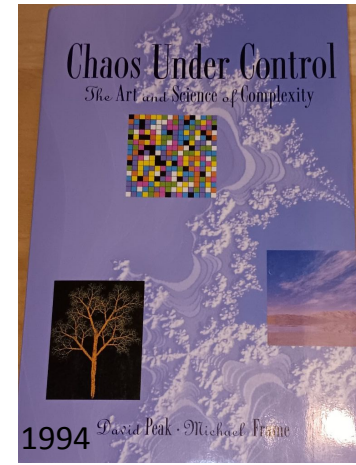
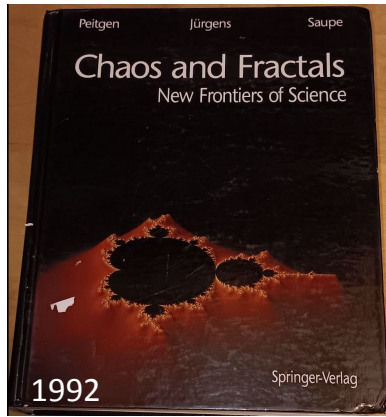
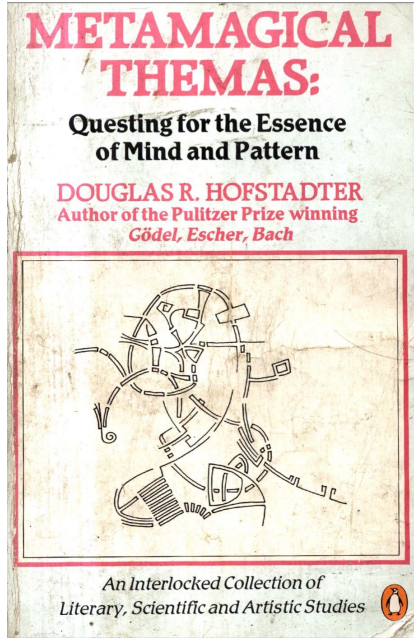




Finis



Key resources



The book was published in 1985.
The article pertinent to chaos was published in Scientific American in November, 1981

References

- For a general summary of the topic, consider these resources
 - J. Gleick, *Chaos*, Penguin 1987
 - Various articles in Wikipedia with titles like Chaos, Dynamical Systems, Strange Attractors
- The following are more technical but don't require calculus
 - Chaos Under Control: The Art and Science of Complexity by Peak and Frame
 - <https://www.amazon.com/Chaos-Under-Control-Science-Complexity/dp/0716724294>
 - Attracted by (STRANGE) Attractors An illustrated guided tour from well known to unknown attractors Josef Böhm (ACDCA and DERIVE & TI-CAS User Group) http://time2012.ut.ee/index_files/proceedings/present/P12paper.pdf
 - Chapter 16 of Metamagical Themas by Hofstadter <https://archive.org/details/MetamagicalThemas>
 - <http://hypertextbook.com/chaos/>
 - <http://math.bu.edu/DYSYS/applets/nonlinear-web.html>
 - <http://www.stsci.edu/~lbradley/seminar/chaos.html>
 - <http://hypertextbook.com/chaos/universality/>
 - For a variety of applets (requiring Java) go here <http://math.bu.edu/DYSYS/applets/>
 - Much more complex presentations
 - Robert May 1976 [Simple mathematical models with very complicated dynamics](#) Nature 261 p. 459
 - Steve Strogatz lectures 19-21 of the series MAE5790 on Youtube. https://www.youtube.com/watch?v=pqJL_K2sS3Y&t=2879s
 - Feigenbaum speaking; <https://www.youtube.com/watch?v=h6mdopJLqf8>
 - Paul Horowitz Chaos circuit <https://www.youtube.com/watch?v=DBteowmSN8g>
 - *Nonlinear Dynamics and Chaos* by Strogatz <https://www.amazon.com/Nonlinear-Dynamics-Chaos-Applications-Nonlinearity/dp/0738204536>
 - To build a circuit that produces bifurcations, consider this reference http://www.cmp.caltech.edu/~mcc/Chaos_Course/Chua/Chua.html
 - Mandelbrot videos are everywhere. Here's an interesting one https://www.youtube.com/watch?v=9gk_8mQuerg
 - These authors have a number of books available in electronic form
 - **Chaos and Fractals: New Frontiers of Science, Edition 2**
 - Heinz-Otto Peitgen Hartmut Jürgens Dietmar Saupe
 - Springer Science & Business Media

References to work on the relation of cardiac emergencies to chaos theory

- **From Clocks to Chaos: The Rhythms of Life** Leon Glass and Michael C. Mackey 1988
https://www.ijstor.org/stable/j.ctv173dzwt?turn_away=true
- [Non-linear dynamics for clinicians: chaos theory, fractals, and complexity at the bedside - The Lancet](#)
- [Fascinating rhythm: A primer on chaos theory and its application to cardiology - ScienceDirect](#)
- [Chaos Theory, Heart Rate Variability, and Arrhythmic Mortality | Circulation.](#)
- [What is Chaos Theory? – Fractal Foundation](#)
- [Chaos Theory, Heart Rate Variability, and Arrhythmic Mortality | Circulation.](#)

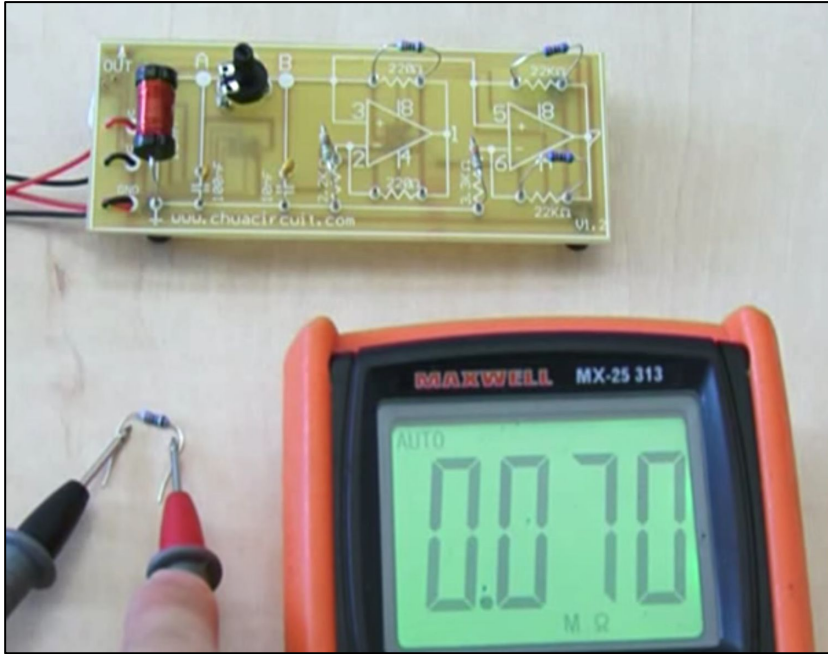
Additional examples

Fractal dimensions

- A **fractal dimension** captures the **complexity** in a **pattern**.
- A **fractal** pattern changes with the **scale** at which it is measured.
- A coastline's measured length changes with the length of the measuring stick used.
- “In some sense, fractal geometry is first and foremost a new ‘language’ used to describe, model and analyze the complex forms found in nature.” Heinz-Otto Peitgen Hartmut Jürgens Dietmar Saupe **Chaos and Fractals** New Frontiers of Science Second Edition

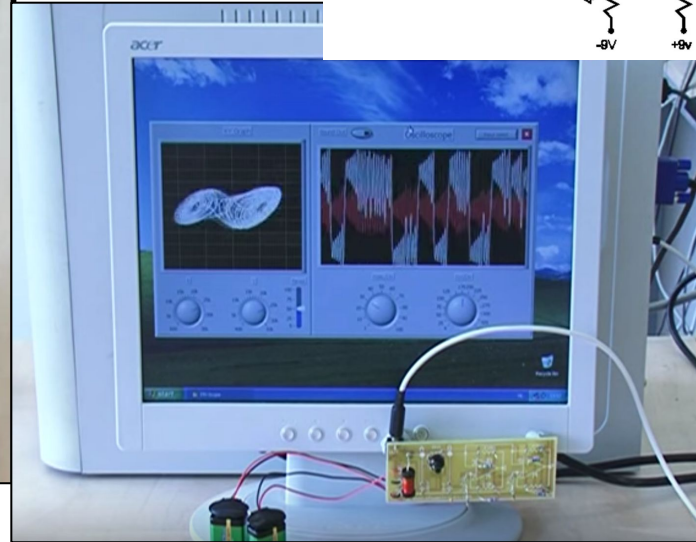
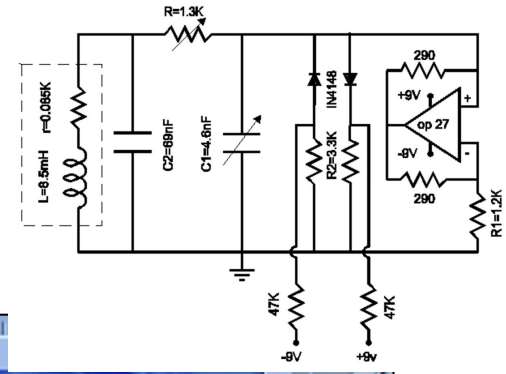


Description of a Chua circuit kit



<https://www.youtube.com/watch?v=bxQr8qI0Hz8>

<https://inst.eecs.berkeley.edu/~ee129/sp10/handouts/ChuasCircuitForHighSchoolStudents-PREPRINT.pdf>

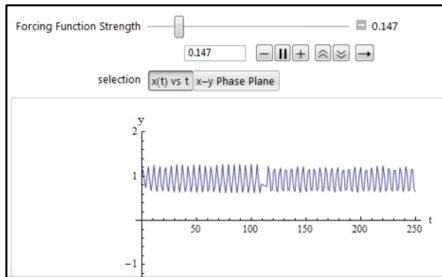


G. Gandhi, Gy. Csereny, J. Zbrozek and T. Roska: "Anyone can build Chua's circuit: Hands-on-Experience with Chaos Theory for High School students", International Journal of Bifurcation and Chaos, Vol. 19, No. 4 (2009) pp. 1113-1125.

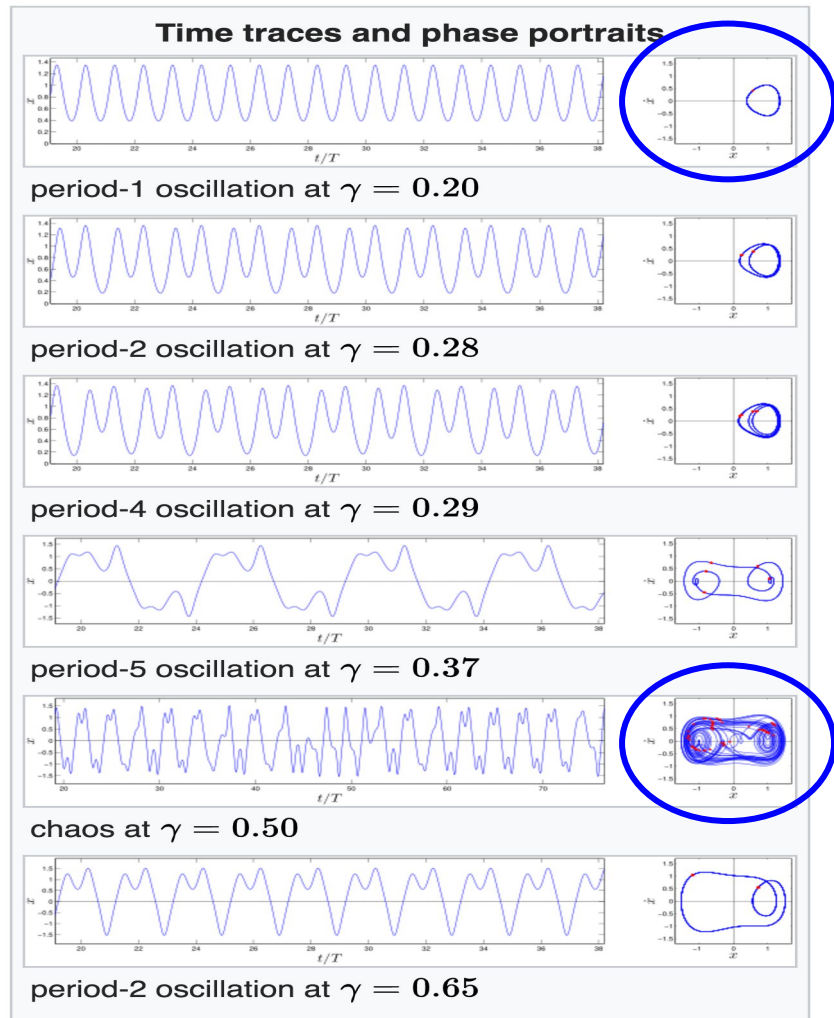
A practical example: Duffing's equation

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$$

- Duffing studied the behavior of harmonic oscillators with a non-linear component
- He derived the equation above to explain behavior of vacuum tubes
- This models, for example, a spring pendulum whose spring's stiffness does not exactly obey Hooke's law
- Periodic behavior is found for a wide range of driving force amplitudes (measured as γ) except for the case of $\gamma=0.5$, where things are chaotic. A slightly higher value restores periodic motion.

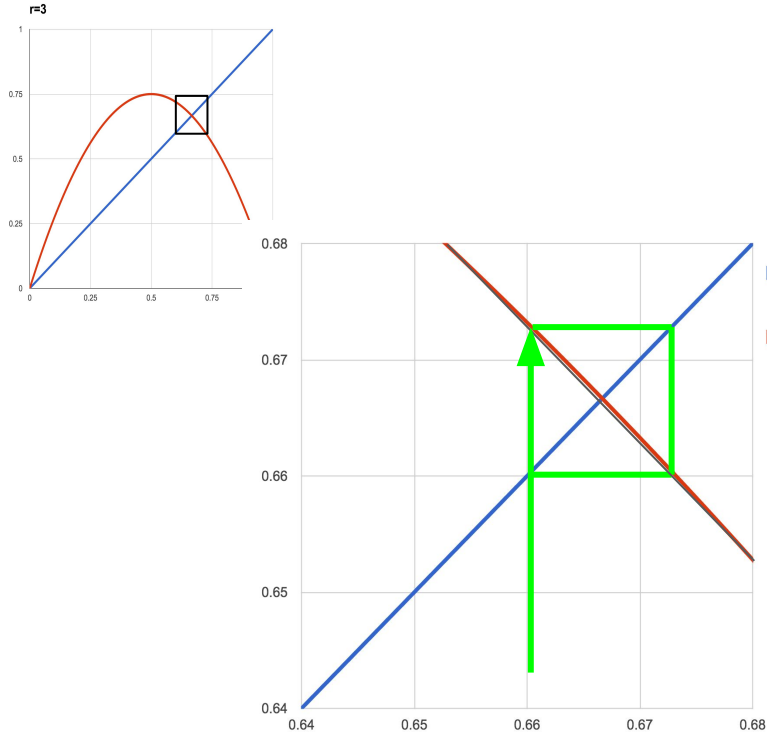


https://www.youtube.com/watch?v=N8x_HluzDII



When $r=3$ we see the first evidence of bifurcation (period 2)

- When the angle of intersection of the line and curve becomes 90 deg, we get a stable 'orbit' with 2 values
- The intersection of the line and curve can occur at a 90° angle (such as for $r=3$) at the same place
- In this case, we can expect 2 x values to persist



We need to compute the point at which the slope of the curve and the slope of the line are perpendicular.

The line slope is 1

The curve slope is the derivative of the parabola function

$$y = rx(1-x) \quad y' = r - 2rx$$

y' is perpendicular to the line when

$$-\frac{1}{y'} = 1 \Rightarrow -\frac{1}{r-2rx} = 1$$

$$-1 = r - 2rx = r(1-2x)$$

$$r(2x-1) = 1 \text{ to assure the curve slope} = 1$$

But to have this happen where curves cross

$$rx(1-x) = x$$

$$r(1-x) = 1$$

So

$$r(2x-1) = r(1-x)$$

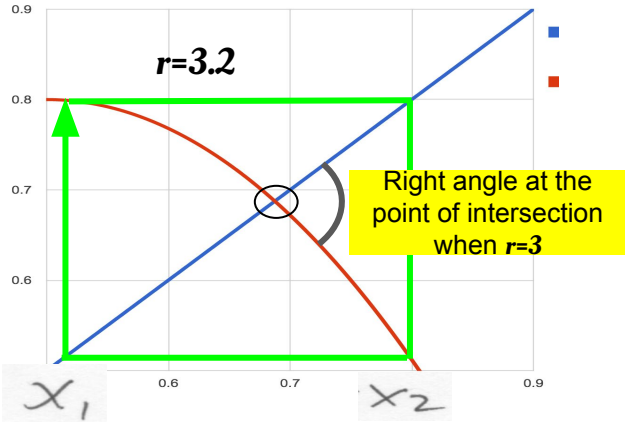
$$2x-1 = 1-x$$

$$3x = 2 \Rightarrow x = \frac{2}{3}$$

which means $r = 3$

Calculating the x boundaries for period 2

- When the r is big enough, we get a stable 'orbit' with 2 values shown as x_1 and x_2
- The following equations define how we get two values: the output of the first iteration (x_2) yields a value that leads back to x_1
- Thus, the results oscillate between the 2 values
- Solving requires dealing with a cubic equation; we get three values



$$f(x_1) = x_2 = rx_1(1-x_1)$$

$$f(x_2) = x_1 = r \underbrace{rx_1(1-x_1)}_{f_1} \underbrace{[1-rx_1(1-x_1)]}_{f_1}$$

$$x_1 = r^2 x(1-x) [1-rx+rx^2]$$

$$r^2(1-x)[1-rx+rx^2] - 1 = 0 \quad \text{Cubic Eq.}$$

$r=3.2$		
r	x_0	x_1
4		0.10
5	1	0.29
6	2	0.66
7	3	0.72
8	4	0.64
9	5	0.74
10002	9997	0.79946
10003	9998	0.51304
10004	9999	0.79946
10005	10000	0.51304
10006	10001	0.79946
10007	10002	0.51304
10008	10003	0.79946
10009	10004	0.51304

- We can use this cubic equation to do an **example** for $r=3.2$
- We get three values
- The first 2 show the two x values that are solutions for $r=3.2$

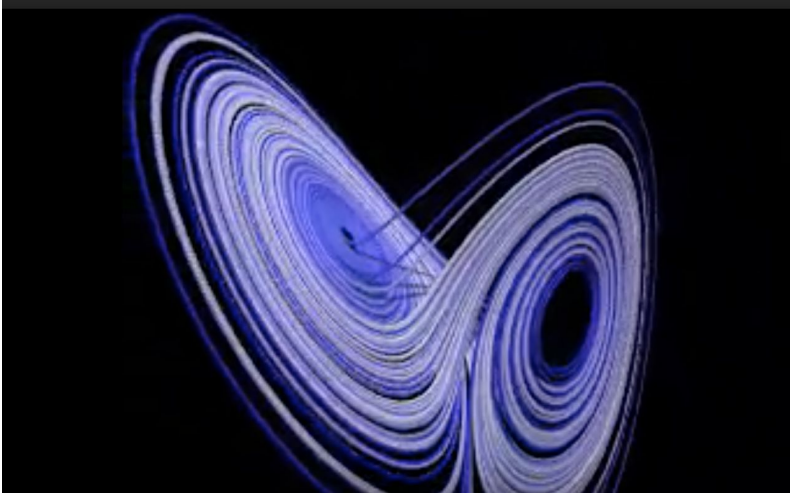
$$x = 0.513045$$

$$x = 0.799455$$

- The third solution shows the location of the intersection of the two curves (at the location of the circle in the graph) $x = 0.6875$

Note that the equation is also cubic in r . What value of r results in a single value of x ? Hint: look at a particular previous graphic in this presentation. From it you can get the solution without doing any math, then check your answer.

A 3d look at the phase diagram of the Lorenz attractor



[lorenz .m4v](#)

